NOTE ON PREVALUATION DOMAINS

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ABSTRACT. We study the structure of one-dimensional prenoramal domain and show the theorem of independence of prevaluation domains.

1. INTRODUCTION

The author intoroduced a concept of prevaluation rings in [6] for the study of the properties $Pic(R) \cong Pic(R[X, X^{-1}])$ and the seminormality due to Traverso([5]). The purpose of this short note is to show the structure of one-dimensional prenormal domain. In particular, if $R = (R, \mathfrak{m}, k)$ is local, $v(\mathfrak{m}) = \dim_k \bar{R}/\mathfrak{m}\bar{R}$.

2. DEFINITIONS AND NOTATIONS

We mean by a ring a commutative ring with identity. The undefined terminology is, in general, the same as that in [4] or [3].

Notation 2.1. (2.1.1) *$\bar{R}$ denotes the derived normal ring of R.*
(2.1.2) *$Q(R)$ denotes the field of quotients of R.*
(2.1.3) *$\kappa(\mathfrak{p})$ denotes the residue field of $R_{\mathfrak{p}}$.*
(2.1.4) *$X^1(R)$ denotes the prime ideals of height one in R.*
(2.1.5) *$\ell_R(M) = \ell(M)$ denotes the length of an R-module M.*
(2.1.6) *$(R, \mathfrak{m}, k)$ denotes a quasi-local ring with residue field k.*

Definition 2.2. ([6]) *Let R be an integral domain with $Q(R) = K$. Then R is said to be a prevaluation domain of K, then either $x \in R$ or $x^{-1} \in \bar{R}$. Inparticular, R is said to be a discrete prevaluation domain if $\bar{R}$ is noetherian.*

Definition 2.3. ([1]) *Let $(R, \mathfrak{m})$ be a local domain of dimension one and let $\bar{\mathfrak{n}}$ be the Jacobson radical of $\bar{R}$. Then R is said to be a weak discrete valuation domain if we have $\mathfrak{m} = \bar{\mathfrak{n}}$ in the set-theoretical sense.*

Definition 2.4. *An integral domain R is called a pre-Krull domain if the following two conditions are satisfied;*
(2.4.1) *If $\mathfrak{p} \in X^1(R)$, then $R_{\mathfrak{p}}$ is a discrete prevaluation domain.*
(2.4.2) *Any non-zero principal ideal of R is the intersection of a finite number of height one.*
The condition (2.4.2) above is equivalent to the following two conditions:
(2.4.2a) *Any principal ideal of R has only a finite number of prime divisor of height one.*
(2.4.2b) *$R = \cap_{\mathfrak{p}} R_{\mathfrak{p}}$, where $\mathfrak{p} \in X^1(R)$*

Definition 2.5. *An integral domain R is called a weak Krull domain if the following two conditions are satisfied:*
(2.5.1) *If $\mathfrak{p} \in X^1(R)$, then $R_{\mathfrak{p}}$ is a weak discrete valuation domain.*

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(2.5.2) *Any non-zero principal ideal of R is the intersection of a finite number of primary ideals of height one.*

Definition 2.6. ([2]) *A ring R is a Mori ring if it is reduced and $\bar{R}$ is finite over R.*

Definition 2.7. ([6]) *Let R be a ring, T an overring of R integral over R. We set*

$$R_T^\natural = \{x \in T \mid x \in R_\mathfrak{p} + \mathfrak{q}T_\mathfrak{p}, \forall \mathfrak{q} \in Spec(T), \forall \mathfrak{p} = \mathfrak{q} \cap R\}$$

$R_T^\natural$ is called the prenormalization of R in T, and if $R = R_T^\natural$, then R is called prenormal in T. If $T = \bar{R}$, we put $R^\natural = R_T^\natural$, and R is called prenormal if $R = R^\natural$.

Definition 2.8. ([5]) *Let R be a ring, T an overring of R integral over R. We set*

$$R_T^+ = \{x \in T \mid x \in R_\mathfrak{p} + J(T_\mathfrak{p}), \forall \mathfrak{p} \in Spec(R)\}$$

where $J(T_\mathfrak{p})$ is the Jacobson radical of $T_\mathfrak{p}$. R_T^+ is called the seminormalization of R in T, and if $R = R_T^+$, then R is called seminormal in T. If $T = \bar{R}$, we put $R^+ = R_T^+$, and R is called seminormal if $R = R^+$.

Definition 2.9. *Then R_T^+ is the largest ring R^* between R and T satisfying the following properties:*

(2.9.1) *For every $\mathfrak{p} \in Spec(R)$, there is a unique prime ideal $\mathfrak{p}^*$ of R^* lying over $\mathfrak{p}$ and $\mathfrak{p}^*$ satisfies $\kappa(\mathfrak{p}^*) = \kappa(\mathfrak{p})$.*

Definition 2.10. *Let R be a reduced ring. R is said to be quasinormal if canonical homomorphism of the Picard groups $Pic(R) \longrightarrow Pic(R[X, X^{-1}])$ is an isomorphism.*

Definition 2.11. *Let $(R, \mathfrak{m}, k)$ be a Macaulay local domain of dimension d. The type of R, $r(R)$, is defined by*

$$r(R) = \dim_k Ext_R^d(k, R)$$

3. Prevaluation domains

We begin with some straight forward observations:

Proposition 3.1. (3.1.1) *Let R be a prevaluation domain of K and let T be an integral domain such that $R \subseteq T \subseteq K$. Then T is a prevaluation domain.*

(3.1.2) *A normal prevaluation domain is a valuation domain. In particular, If R is a prevaluation domain, then $\bar{R}$ is a valuation domain, hence R is a quasi-local domain.*

(3.1.3) *There exists a prevaluation domain, which is not a valuation domain.*

Example: Let $K \subseteq L$ be fields and L is finite algebraic over K. Then $K + X \cdot L[[X]]$ is the required example.

(3.1.4) *A noetherian prevaluation domain is a discrete parevaluation domain by the theorem of Krull-Akizuki([4](33.2))*

(3.1.5) *There exists a discrete prevaluation domain which is not noetherian.*

Example: Let $K \subseteq L$ be fields and L is infinite algebraic over K. Then $K + X \cdot L[[X]]$ is the required example.

The following proposition is an immediate consequence of Definition(1.2), too.

Proposition 3.2. *Let R be an integral domain with $Q(R) = K$. Then the following statements are equivalent.*

(3.2.1) *R is a prevaluation domain of K.*

(3.2.2) *$\forall a, b \in R$, $a \in b\bar{R}$ or $b \in a\bar{R}$*

(3.2.3) *$\bar{R}$ is a valuation domain and the maximal ideal of $\bar{R}$ is set-thoretically equal to the maximal ideal of R.*

(3.2.4) *$\bar{R}$ is a valuation domain and , for any prime ideal $\mathfrak{p}$ of R, a maximal ideal of $\bar{R}_{\mathfrak{p}}$ is set-theoretically equal to $\mathfrak{p}$.*

Proof. (3.2.1) $\Longleftrightarrow$ (3.2.2) *is nothing but a restatement of the definition.*

(3.2.1) $\Longrightarrow$ (3.2.3): *By* (3.1.2) *$\bar{R} = (\bar{R}, \bar{\mathfrak{m}})$ is a valuation domain. Then R is a quasi-local domain $(R, \mathfrak{m})$. It is clear that $\mathfrak{m}$ is contained in $\bar{\mathfrak{m}}$. Take an element of $\mathfrak{m}$, say x. Since $\bar{R}$ is a valuation domain, x^{-1} is not in $\bar{R}$. Hence $x \in R \cap \bar{\mathfrak{m}} = \mathfrak{m}$.*

(3.2.3) $\Longrightarrow$ (3.2.1): *Suppose $x-1$ is not in $\bar{R}$. Since $\bar{R} = (\bar{R}, \mathfrak{m})$ is a prevaluation domain, x is in $\mathfrak{m}$. Hence $x \in \bar{\mathfrak{m}} = \mathfrak{m} \subset R$.*

(3.2.3) $\Longleftrightarrow$ (2.2.4): *This is straight forward by* (3.6.2).

Corollary 3.3. *A noetherian prevaluation domain is a unibranched weak discrete valuation domain.*

Corollary 3.4. *A prevaluation domain is seminormal.*

Proof. Let $\mathfrak{p}$ be any prime ideal of R and let $J(\bar{R}_{\mathfrak{p}})$ be the Jacobson radical of $\bar{R}_{\mathfrak{p}}$. Since $R_{\mathfrak{p}}$ is a prevaluation domain by (3.1.1), $J(\bar{R}_{\mathfrak{p}}){=}J(R_{\mathfrak{p}})$ by (3.2). Hence $J(\bar{R}_{\mathfrak{p}}) \subseteq R_{\mathfrak{p}}$. Thus $R^+ = \cap[R_{\mathfrak{p}} + J(\bar{R}_{\mathfrak{p}})] = \cap R_{\mathfrak{p}} = R$.

Corollary 3.5. *A prenormal domain is seminormal.*

Proof. It is obvious by the preceding proof.

Proposition 3.6. *Let R be a prevaluation domain of K and let $\mathfrak{p}$ be a prime ideal of $R/\mathfrak{p}$. Then, the following statements holds.*

(3.6.1) *If a is an element of R which is not in $\mathfrak{p}$, then $\mathfrak{p}$ is contained in aR.*

(3.6.2) *$\mathfrak{p}$ is set-theoretically equal to $\mathfrak{p}R_{\mathfrak{p}}$.*

(3.6.3) *$R/\mathfrak{p}$ is a prevaluation domain.*

Proof. (3.6.1): Let x be any arbitrary element of $\mathfrak{p}$. Suppose x is not an element of aR. By (3.2), a is an element of $x\bar{R}$, namely a/x is an element of $\bar{R}$. Therefore we have

$$(a/x)^m + c_1(a/x)^{m-1} + \cdots + c_m = 0 \qquad (c_1, ..., c_m \in R).$$

Hence a^m is an element of xR. Since xR is contained in $\mathfrak{p}$, a is an element of $\mathfrak{p}$, a contradiction.

(3.6.2): Let x be any element of $\mathfrak{p}R_{\mathfrak{p}}$. Then there is an element s of R which is not in $\mathfrak{p}$ and such that $t = sx$ is in $\mathfrak{p}$. Since s is not in $\mathfrak{p}$, $tR \subset sR$ by (3.6.1), and we have $x \in R$. Moreover, since sx is in $\mathfrak{p}$, x is an element of $\mathfrak{p}$.

(3.6.3): Let a, b be elements of R and denote by $\bar{a}$ the image of a in $R/\mathfrak{p}$. Since either $a \in b\bar{R}$ or $b \in a\bar{R}$, either $\bar{a} \in \bar{b}\widetilde{(R/\mathfrak{p})}$ or $\bar{b} \in \bar{a}\widetilde{(R/\mathfrak{p})} = \bar{a}(\bar{R}/\mathfrak{p})$.

Proposition 3.7. *Let $(R, \mathfrak{m}, k)$ be a noetherian prevaluation domain with $\bar{R} = (\bar{R}, \bar{\mathfrak{m}}, \bar{k})$. Then $v(\bar{\mathfrak{m}}) = [\bar{k} : k]$.*

Proof. Let x be an element of R such that $x\bar{R} = \bar{\mathfrak{m}}$ and let the y_j's be elements of R such that $\bar{\mathfrak{m}} = xR + y_1R + \cdots + y_{t-1}R$ where $v(\bar{m}) = t$. Since y_j is also an element of $\bar{\mathfrak{m}} = x\bar{R}$, $y_j = xu_j$, where u_j is an element of $\bar{R}$ and is a unit of $\bar{R}$ by (3.2). Thus the u_j(*modulo* $\mathfrak{m}$)'s are the basis of $\bar{k}$ over k. Therefore $t = [\bar{k} : k]$.

Corollary 3.8. *Let $(R, \mathfrak{m}, k)$ be a noetherian prevaluation domain with $r(R) = s$ and put $\bar{R} = (\bar{R}, \bar{\mathfrak{m}}, \bar{k})$. Then we have $[\bar{k} : k] = s + 1$.*

Proof. Let x be an element of R such that $x\bar{R} = \bar{\mathfrak{m}}$. Hence $T = (T, \mathfrak{n}) = (R/xR,\ \mathfrak{m}/xR)$ is a local ring of dimension zero with $r(T) = s$. Therefore $\ell_T(0 : \mathfrak{n}) = s$. On the other hand, we can see that $(0 : \mathfrak{n})_T = \mathfrak{n}$ by the proof of (3.7). Thus we see that $v(\mathfrak{n}) = s$, i.e., $v(\mathfrak{m}) = s + 1$. Hence $[\bar{k} : k] = s + 1$.

Then we have

Theorem 3.9. *Let $(R, \mathfrak{m}.k)$ be a Mori local domain of dimension one which is not normal. The following statements are equivalent:*

(3.9.1) *R is prenormal and $v(\mathfrak{m}) = 2$.*

(3.9.2) *R is prenormal and Gorenstein.*

(3.9.3) *R is prenormal and $r(R) = 1$.*

(3.9.4) *R is a prevaluation domain and $\dim_k \bar{R}/\mathfrak{m}\bar{R} = 2$.*

(3.9.5) *R is a prevaluation domain and Hilbert polynomial of R is $f(n) = 2n + 1$.*

(3.9.6) *$e(R) = 2$ and $R/\mathfrak{m}R$ is an itegral domain.*

(3.9.7) *$e(R) = 2$ and $gr(R)$ is an itegral domain.*

If moreover R contains a field the above are equivalent to:

(3.9.8) *The completion of R is $\hat{R} = k + X \cdot k(u)[[X]]$. where $k(u)$ is a field extension of degree 2 of k and X is a transcendental element over k.*

Proof. It is clear that (3.9.1)$\Longleftrightarrow$(3.9.2)$\Longleftrightarrow$(3.9.3)$\Longleftrightarrow$(3.9.4)$\Longleftrightarrow$(3.9.5)$\Longleftrightarrow$(3.9.6)$\Longleftrightarrow$(3.9.7).

(3.9.4)$\Longleftrightarrow$(3.9.8): This is straight forward by the structure theorem of complete local rings([4],(31.1)).

Corollary 3.10. *Let $(R, \mathfrak{m}, k)$ be a Mori domain of dimension one which is not normal. The following statements are equivalent:*

(3.10.1) *R is prenormal and $v(\mathfrak{m}) = t + 1$*

(3.10.2) *R is prenormal and $r(R) = t$.*

(3.10.3) *R is a prevaluation domain and $\dim_k \bar{R}/\mathfrak{m}\bar{R} = t + 1$.*

(3.10.4) *R is a prevaluation domain and the Hilbert polynomial of R is $f(n) = (t + 1)n + 1$.*

(3.10.5) *$e(R) = t + 1$ and $\bar{R}/\mathfrak{m}\bar{R}$ is an integral domain.*

(3.10.6) *$e(R) = t + 1$ and $gr(R)$ is an integral domain.*

If moreover R contains a field the above are equivalent to:

(3.10.7) *The completion of R is $\hat{R} = k + X \cdot L[[X]]$, where L is a field extension of degree $t + 1$ of k and X is a transcendental element over k.*

Finally we have add here next remark on prevaluation domains.

Corollary 3.11. *Let a domain R be the intersection $W_1 \cap W_2 \cap \cdots \cap W_n$, where the W_j 's are prevaluation domains between R and $Q(R)$. Then W_j has the form $R_{\mathfrak{p}_j}$ for a suitable $\mathfrak{p}_j \in Spec(R)$.*

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